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Joint Admission Examination for Macao Four Higher Education Institutions

(Languages and Mathematics)

2023

2023 Examination Paper and Suggested Answer

Mathematics Standard Paper

第一部份 選擇題。請選出每題之

1. $M=\{x \mid x^2-2x-8\geq 0\}$ $N=\{x \mid 0<x<6\}$ $M\cap N=($ $)$

A. $[-2,4]$ B. $[-2,0)$ C. $(0,4]$ D. $(0,6)$ E. $[4,6)$

2. $f(x)$ x^2-x-6 $3x-2$ $f(3)=($ $)$

A. -2 B. 0 C. 3 D. 7 E. 9

3. $\log_9 125\times\log_{12}17\times\log_{25}3\times\log_{17}12=($ $)$

A. $\log_{17}3$ B. $\frac{1}{2}$ C. $\frac{3}{4}$ D. $\log_3 35$ E. $\log_{17}12$

4. $x^2-3x+4\sqrt{x^2-3x}=12$ ( )

A. $\{-1\}$ B. $\{2,-6\}$ C. $\{-1,4\}$ D. $\{4\}$ E. $\{3\}$

5. a $4a^2x^2+2(a+3)x+9=0$ $a=($ $)$

A. $\frac{3}{5}$ B. -1 $\frac{3}{2}$ C. $\frac{3}{2}$ D. $-\frac{3}{7}$ $\frac{3}{5}$ E.

6. $\left(2\sqrt{x}-\frac{1}{\sqrt{x}}\right)^6$ ( )

A. -8 B. 8 C. -160 D. 160 E. 1

7. $f(x)=ax^2+4x+1$ $a\in\mathbb{R}$ $(2,4)$ a ( )

A. $\left[-\frac{1}{2},0\right)$ B. $\left(0,\frac{1}{2}\right]$ C. $\left[-\frac{1}{2},\frac{1}{2}\right]$ D. $\left[-\frac{1}{2},\infty\right)$ E. $\left[\frac{1}{2},\infty\right)$

8. $f(x)=\begin{cases}\log_2 x, & 0<x\leq 4\\ x^2-8x+17, & x>4\end{cases}$ $f(\frac{1}{2}-3|x|)+f(5)>0$ ( )

A. $-\frac{1}{12}<x<\frac{1}{12}$ B. $-\frac{1}{6}<x<\frac{1}{6}$ C. $-\frac{1}{4}<x<\frac{1}{4}$

D. $-\frac{1}{3}<x<\frac{1}{3}$ E. $-\frac{1}{2}<x<\frac{1}{2}$

9. 3 8 5 2

( )

A. $\frac{2}{3}$ B. $\frac{3}{2}$ C. 1 D. $\frac{16}{27}$ E. $\frac{32}{27}$

10. 7 80 16 26 34 ( )

A. -6 B. -82 C. -88 D. -198 E. -204

11. $A(3,-8)$ $B(-7,4)$ AB $3x-4y+14=0$ ( )

A. $4x+3y+14=0$ B. $3x+4y+14=0$ C. $3x-4y-14=0$

D. $4x-3y+14=0$ E. $4x+3y-14=0$

12. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\ (a,b>0)$ 3 $\frac{b^2+2}{a}$ ( )

A. 2 B. $2\sqrt{2}$ C. $2\sqrt{3}$ D. 4 E. 8

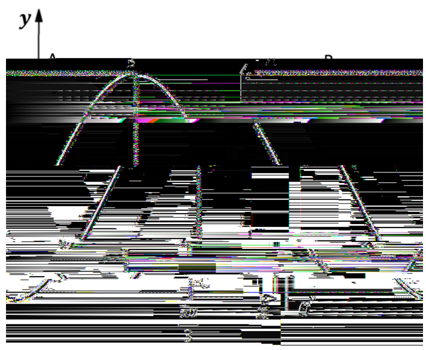
13. A B $\sin A=\frac{2}{5}$ $\sin B=\frac{4}{5}$ $\sin(A+B)=($ $)$

A. $\frac{-6-4\sqrt{21}}{25}$ B. $\frac{13}{25}$ C. $\frac{18}{25}$

D. $\frac{-12-2\sqrt{21}}{25}$ E. $\frac{12+2\sqrt{21}}{25}$

14. $y=a\sin(x-\frac{\pi}{6})+b$ a b

( )



A. $a=-4$ $b=4$ B. $a=-2$ $b=2$

C. $a=2$ $b=-2$ D. $a=4$ $b=-4$

E.

15. $A(-2,3)$ O $90°$ B

第二部份 解答题。

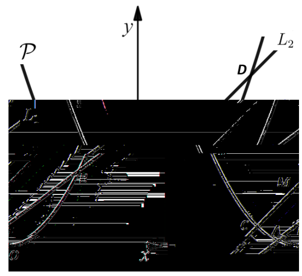
1. $\frac{1}{4}$

(a) (3)

(b) (2)

(c) (3)

2. $\mathcal{P}\colon x^2 = 4y$ F F $\frac{3}{4}$ L_1 $\mathcal{P}$ A B 1 L_2 $\mathcal{P}$ C D y M



(a) F (2)

(b) AB (3)

(c) $|DM| = 3|CM|$ CD (3)

3. $S_n = 3^{n+1} - 2k$ $\{a_n\}_{n\geq 1}$ n $k \in \mathbb{R}$

(a) k a_n (3)

(b) $b_n = \dfrac{1}{a_n} + \log_2 a_n$ b_n n T_n (3)

(c) $c_n = \dfrac{2}{a_n}$ $f(n) = -5c_n^2 + c_n$ n (2)

4. $f(x) = \sqrt{3}\sin(2wx) - 2\cos^2(wx)$ 3π

(a) $f(x)$ (4)

(b) $\triangle ABC$ $f(C) = 0$ $2\sin^2 B = \cos B + \cos(A - C)$ $\sin A$ (4)

5. x, y $\begin{cases} 3x + 2y - 13 \geq 0 \\ x \leq 5 \\ 2x - 2y + 3 \geq 0 \end{cases}$

(a) (2)

(b) $z = \dfrac{y}{x}$ z (3)

(c) $t = x^2 + y^2$ t (3)

<u>第一部份</u> 選擇題。

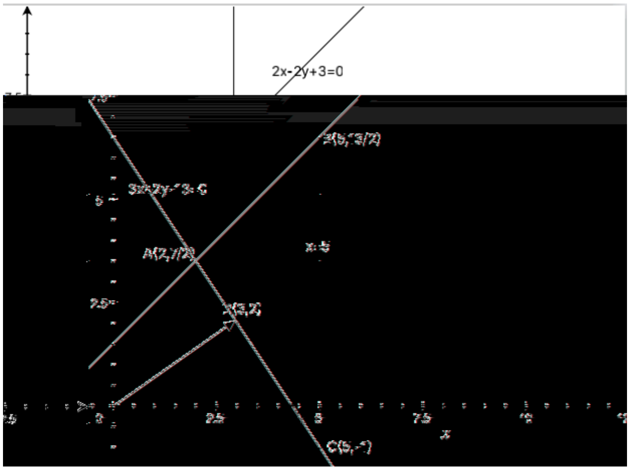
1	E
2	D
3	C
4	C
5	D
6	C
7	D
8	A
9	E
10	B
11	A
12	E
13	A
14	B
15	E

(b) $f(C) = 2\sin(\frac{2}{3}C - \frac{\pi}{6}) - 1 = 0$ $\sin(\frac{2}{3}C - \frac{\pi}{6}) = \frac{1}{2}$ $-\frac{\pi}{6} < \frac{2}{3}C - \frac{\pi}{6} < \frac{\pi}{2}$

$\frac{2}{3}C - \frac{\pi}{6} = \frac{\pi}{6}$ $C = \frac{\pi}{2}$ $A + B = \frac{\pi}{2}$ $2\sin^2 B = \cos B + \cos(A - \frac{\pi}{2})$

$2\sin^2 B = \sin A + \sin A$ $\sin^2 B = \sin A$ $1 - \sin^2 A = \sin A$ $\sin A = \frac{\sqrt{5}-1}{2}$

5. (a)



(b) y/x $P(x, y)$ $A(2, 7/2)$,$B(5, 13/2)$

$C(5, -1)$ P z C A

$-1/5 \leq z \leq 7/4$

(c) t $P(x, y)$ D $3x + 2y - 13 = 0$

OD OD $2/3$ $y = 2x/3$ OD

$3x + 2y - 13 = 0$ $(x, y) = (3, 2)$ t $(x, y) = (3, 2)$ 13

Part I Multiple choice questions. Choose the for each question.

1. Let $M = \{x \mid x^2 - 2x - 8 \geq 0\}$ and $N = \{x \mid 0 < x < 6\}$, then $M \cap N = (\quad)$.

   A. $[-2, 4]$ B. $[-2, 0)$ C. $(0, 4]$ D. $(0, 6)$ E. $[4, 6)$

2. If we divide the polynomial $f(x)$ by $x^2 - x - 6$ and the remainder is $3x - 2$, then $f(3) = (\quad)$.

   A. -2 B. 0 C. 3 D. 7 E. 9

3. $\log_9 125 \times \log_{12} 17 \times \log_{25} 3 \times \log_{17} 12 = (\quad)$.

   A. $\log_{17} 3$ B. $\frac{1}{2}$ C. $\frac{3}{4}$ D. $\log_3 35$ E. $\log_{17} 12$

4. The set of solutions for the equation $x^2 - 3x + 4\sqrt{x^2 - 3x} = 12$ is $(\quad)$.

   A. $\{-1\}$ B. $\{2, -6\}$ C. $\{-1, 4\}$ D. $\{4\}$ E. $\{3\}$

5. Let a be a constant and suppose the quadratic equation $4a^2x^2 + 2(a + 3)x + 9 = 0$ has exactly one real solution. Then $a = (\quad)$.

   A. $\frac{3}{5}$ B. -1 or $\frac{3}{2}$ C. $\frac{3}{2}$
   D. $-\frac{3}{7}$ or $\frac{3}{5}$ E. any real number

6. The constant term in the expansion of $\left(2\sqrt{x} - \frac{1}{\sqrt{x}}\right)^6$ is $(\quad)$.

   A. -8 B. 8 C. -160 D. 160 E. 1

7. The function $f(x) = ax^2 + 4x + 1$ ($a \in \mathbb{R}$ is a constant) is increasing on the open interval $(2, 4)$. Then the range of a is $(\quad)$.

   A. $\left[-\frac{1}{2}, 0\right)$ B. $\left(0, \frac{1}{2}\right]$ C. $\left[-\frac{1}{2}, \frac{1}{2}\right]$ D. $\left[-\frac{1}{2}, \infty\right)$ E. $\left[\frac{1}{2}, \infty\right)$

8. Let $f(x) = \begin{cases} \log_2 x, & 0 < x \leq 4 \\ x^2 - 8x + 17, & x > 4 \end{cases}$. The solution of the inequality $f(\frac{1}{2} - 3|x|) + f(5) > 0$ is $(\quad)$.

   A. $-\frac{1}{12} < x < \frac{1}{12}$ B. $-\frac{1}{6} < x < \frac{1}{6}$ C. $-\frac{1}{4} < x < \frac{1}{4}$

   D. $-\frac{1}{3} < x < \frac{1}{3}$ E. $-\frac{1}{2} < x < \frac{1}{2}$

9. An upright cylindrical water tank has an inner radius of 3 meters and a height of 8 meters, and the current water depth is 5 meters. If a sphere with a radius of 2 meters is placed into the water tank and the sphere is completely immersed in the water, the water level rises by (   ) meters.

A. $\frac{2}{3}$ B. $\frac{3}{2}$ C. 1 D. $\frac{16}{27}$ E. $\frac{32}{27}$

10. In an arithmetic sequence, the 7th term is 80 and the 16th term is 26. Then the 34th term is (   ).

A. -6 B. -82 C. -88 D. -198 E. -204

11. Let A and B be the points $(3, -8)$ and $(-7, 4)$ respectively. An equation of the line passing through the midpoint of AB and perpendicular to $3x - 4y + 14 = 0$ is (   ).

A. $4x + 3y + 14 = 0$ B. $3x + 4y + 14 = 0$ C. $3x - 4y - 14 = 0$

D. $4x - 3y + 14 = 0$ E. $4x + 3y - 14 = 0$

12. If the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\ (a, b > 0)$ is 3, then the minimum value of $\frac{b^2+2}{a}$ is (   ).

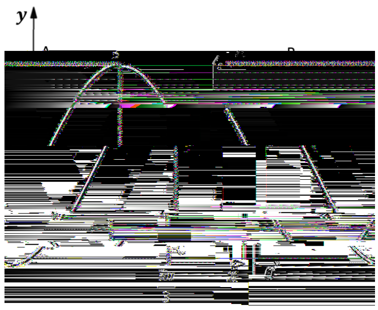
A. 2 B. $2\sqrt{2}$ C. $2\sqrt{3}$ D. 4 E. 8

13. Let A and B be angles in the second quadrant such that $\sin A = \frac{2}{5}$ and $\sin B = \frac{4}{5}$. Then $\sin(A+B) =$ (   ).

A. $\frac{-6-4\sqrt{21}}{25}$ B. $\frac{13}{25}$ C. $\frac{18}{25}$

D. $\frac{-12-2\sqrt{21}}{25}$ E. $\frac{12+2\sqrt{21}}{25}$

14. The right figure shows the graph of $y = a\sin(x - \frac{\pi}{6}) + b$, where a and b are constants. Then (   ).



A. $a = -4$ and $b = 4$ B. $a = -2$ and $b = 2$

C. $a = 2$ and $b = -2$ D. $a = 4$ and $b = -4$

E. none of the above

15. Point $A(-2, 3)$ is rotated $90°$ clockwise about the origin O to get point B. Points C and B are symmetrical about the x-axis. Point C is translated downward three units to get point D. Then the coordinates of point D are (   ).

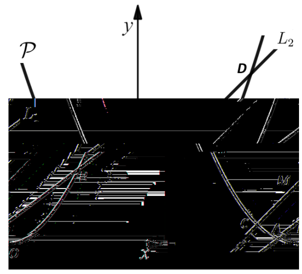
A. $(-3, -1)$ B. $(-3, 0)$ C. $(-4, 0)$ D. $(2, 0)$ E. $(3, -5)$

<u>Part II</u> Problem-solving questions.

1. A coin is unfair that the probability of a head facing up is $\frac{1}{4}$.

   (a) Find the probability of obtaining at most one head facing up in 10 successive tosses. (3 marks)
   (b) Find the probability that the 10th toss will be the first of obtaining head facing up. (2 marks)
   (c) Find the probability that the 10th toss will be the third of obtaining head facing up. (3 marks)

2. In the right figure, the parabola $\mathcal{P} : x^2 = 4y$ has its focus F. The straight line L_1 of slope $\frac{3}{4}$ passing through the focus F intersects the parabola $\mathcal{P}$ at points A and B. Another straight line L_2 of slope 1 intersects the parabola $\mathcal{P}$ at points C and D, and intersects the y-axis at point M.



   (a) Find the coordinates of the focus F. (2 marks)
   (b) Find the length of segment AB. (3 marks)
   (c) If $|DM| = 3|CM|$, find the length of segment CD. (3 marks)

3. Let $S_n = 3^{n+1} - 2k$ be the nth sum of the geometric sequence $\{a_n\}_{n\geq 1}$. Here $k \in \mathbb{R}$ is a constant.

   (a) Find k and a_n. (3 marks)
   (b) Let $b_n = \frac{1}{a_n} + \log_2 a_n$. Find the sum T_n of the first n terms for the sequence b_n. (3 marks)
   (c) Let $c_n = \frac{2}{a_n}$. Find n where $f(n) = -5c_n^2 + c_n$ obtains its maximum value. (2 marks)

4. The minimal positive period of the function $f(x) = \sqrt{3}\sin(2wx) - 2\cos^2(wx)$ is 3π.

   (a) Find the expression of $f(x)$. (4 marks)
   (b) In $\triangle ABC$, if $f(C) = 0$ and $2\sin^2 B = \cos B + \cos(A - C)$, find the value of $\sin A$. (4 marks)

5. Let x, y satisfy $\begin{cases} 3x + 2y - 13 \geq 0 \\ x \leq 5 \\ 2x - 2y + 3 \geq 0 \end{cases}$.

   (a) Sketch the region satisfying the above system of inequalities. (2 marks)
   (b) Let $z = \frac{y}{x}$. Find the range of z. (3 marks)
   (c) Let $t = x^2 + y^2$. Find the minimum value of t. (3 marks)

Suggested Answer

<u>Part I</u> Multiple choice questions.

Question Number	Best Answer
1	E
2	D
3	C
4	C
5	D
6	C
7	D
8	A
9	E
10	B
11	A
12	E
13	A
14	B
15	E

Part II **Problem-solving questions.**

1. (a) If a coin is tossed 10 times consecutively, the probability of getting tails 10 times is $(1-\frac{1}{4})^{10}=(\frac{3}{4})^{10}$. The probability of getting exactly one head in ten consecutive tosses of a coin is ${}_{10}C_1\frac{1}{4}(1-\frac{1}{4})^{10-1}=\frac{5}{2}(\frac{3}{4})^9$. Therefore, the probability of at most one head up is $(\frac{3}{4})^{10}+\frac{5}{2}(\frac{3}{4})^9=\frac{13}{4}\times(\frac{3}{4})^9$.

   (b) The probability that the first 9 tosses are tails and the 10th toss is head is $(1-\frac{1}{4})^{10-1}\times\frac{1}{4}=\frac{3^9}{4^{10}}$.

   (c) The probability of getting exactly two heads in the first 9 tosses is ${}_9C_2(\frac{1}{4})^2(1-\frac{1}{4})^{9-2}$. Therefore, the probability of getting a third head on the tenth toss is ${}_9C_2(\frac{1}{4})^2(1-\frac{1}{4})^{9-2}\times\frac{1}{4}=(\frac{3}{4})^9$.

2. (a) The coordinates of the focus F is $(0,1)$.

   (b) The equation of the straight line L_1 is $y=\frac{3}{4}x+1$. Combining the equations of the straight line L_1 and the parabola $\mathcal{P}$, we can get $x^2=3x+4$. Solving the quadratic equation, one has $x=-1$ or 4. Then the coordinates of points A and B are $A(-1,\frac{1}{4})$ and $B(4,4)$, respectively. Then according to the definition of parabolas, we have $|AB|=\frac{25}{4}$.

   (c) Suppose the equation of the straight line L_2 is $y=x+t$. Suppose the x-coordinates of points C and D are x_1 and x_2, respectively. Combining the equations of the straight line L_2 and the parabola $\mathcal{P}$, we can get $x^2-4x-4t=0$. Using Weda's Theorem, $x_1+x_2=4$. Since $|DM|=3|CM|$, we have $x_2=-3x_1$. Furthermore, we can get $x_1=-2, x_2=6$. Thus, $|CD|=\sqrt{1^2+1}|x_4-x_3|=8\sqrt{2}$.

3. (a) From the question, we get $a_1=S_1=9-2k, a_2=S_2-S_1=27-9=18$ and $a_3=S_3-S_2=54$. Since $\{a_n\}_{n\geq1}$ is a geometric sequence, we have $a_1a_3=a_2^2$, which implies that the first term $a_1=6$, the common ratio $q=3$ and $k=3/2$. Then we can get $a_n=a_1\times q^{n-1}=2\times3^n$.

   (b) Since $b_n=\frac{1}{2\times3^n}+\log_2(2\times3^n)=\frac{1}{2\times3^n}+1+n\log_2 3, T_n=\frac{1}{4}\left(1-\frac{1}{3^n}\right)+n+\frac{n(n+1)}{2}\log_2 3$.

   (c) Since $c_n=3^{-n}$, $f(n)=-5(3^{-n})^2+3^{-n}=-5(\frac{1}{3^n}-\frac{1}{10})^2+\frac{1}{20}$. When $n=2$, the function $f(n)$ obtains its maximum value $\frac{4}{81}$.

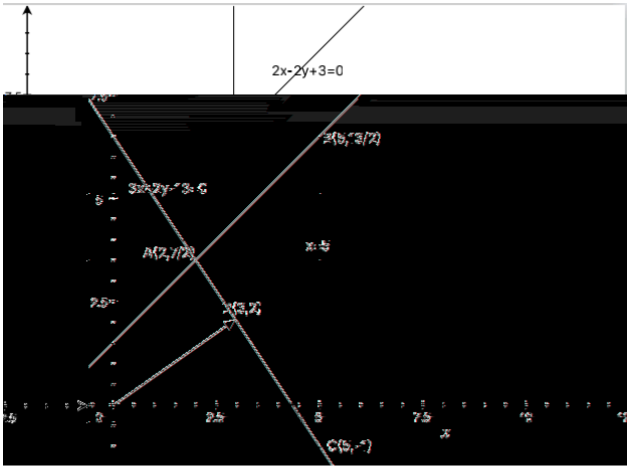
4. (a) By the double-angle formula, $f(x)=\sqrt{3}\sin 2wx-(1+\cos 2wx)$. So $f(x)=2(\frac{\sqrt{3}}{2}\sin 2wx-$

$\frac{1}{2}\cos 2wx) - 1 = 2\sin(2wx - \theta) - 1$, with $\sin\theta = \frac{1}{2}$ and $\cos\theta = \frac{\sqrt{3}}{2}$. Thus $\theta = \frac{\pi}{6} + 2k\pi$. Since the least period of $f(x)$ is $\frac{2\pi}{2w} = 3\pi$, we get $2w = \frac{2}{3}$ and can write $f(x) = 2\sin(\frac{2}{3}x - \frac{\pi}{6}) - 1$.

(b) Since $f(C) = 2\sin(\frac{2}{3}C - \frac{\pi}{6}) - 1 = 0$, we have $\sin(\frac{2}{3}C - \frac{\pi}{6}) = \frac{1}{2}$. By observing that $-\frac{\pi}{6} < \frac{2}{3}C - \frac{\pi}{6} < \frac{\pi}{2}$, we can get $\frac{2}{3}C - \frac{\pi}{6} = \frac{\pi}{6}$. Then $C = \frac{\pi}{2}$ and $A + B = \frac{\pi}{2}$. We have $2\sin^2 B = \cos B + \cos(A - \frac{\pi}{2})$ which implies that $2\sin^2 B = \sin A + \sin A$. Then $1 - \sin^2 A = \sin A$, and $\sin A = \frac{\sqrt{5}-1}{2}$.

5. Answer:

(a)



(b) y/x is the slope of straight line jointing point $P(x, y)$ and the origin. Intersections of the given straight lines are $A(2, 7/2)$, $B(5, 13/2)$ and $C(5, -1)$. When point P varies inside the given region, the minimum value of z can be obtained at point C and the maximum value can be obtained at point A. Then$-1/5 \leq z \leq 7/4$.

(c) t is the square of the distance between point $P(x, y)$ in the given region and the origin. The nearest point to the origin is the point D lying on the line $3x + 2y - 13 = 0$ and OD is perpendicular to this straight line. The slope of the line OD should be $2/3$ and the equation is $y = 2x/3$. Thus, the intersection of lines OD and $3x + 2y - 13 = 0$ is $(x, y) = (3, 2)$. Therefore, t obtains its minimum value 13 at point $(x, y) = (3, 2)$.